# APPROXIMATE CALCULATION OF ST.VENANT EDGE EFFECTS FOR BOUNDARY PROBLEMS IN THE STATICS OF PLATES

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Love [1] considered particular solutions of equations of the three-dimensional theory of elasticity for the basic state of stress. Lur'e, with the aid of a symbolic method, described in [2] and [3] two infinite sequences of particular solutions of the type of St.Venant edge effects. Nevertheless, simplified boundary problems with edge effects are solved for the state of stress only by application of relaxation methods in the three-dimensional theory of elasticity [2] to [4]. Therefore, work on approximate methods of solution of problems with St.Venant edge effects deserves attention.

Asymptotic methods were presented in [5] to [7] for the construction of equations successively determining particular solutions for edge effects in the basic stress state, with asymptotic errors of the order of $a$, $a^2$, $a^3$,.... ($a \to 0$ and is relative plate thickness). Friedrichs and Dressler [6] showed the edge effects with these equations in the case of a free edge, Gol'denveizer [7] considered still other boundary problems. In [8] another variant of the asymptotic method (as $a \to 0$) was applied; it was based on the application of particular solutions known from [2]. In particular, the asymptotic error in Kirchhoff theory was studied in [8] for certain boundary problems. In spite of this, the question of approximate numerical solution of boundary value problems remains insufficiently investigated.

An approximate method of solution for boundary value problems is presented below based on the expanded state of stress and on edge effects expanded in a series of Legendre polynomials along the coordinate normal to the middle surface. As a concrete example, a strip is considered, clamped at the edges and bent under a uniformly distributed load $p$ on both faces. Numerical results are obtained, retaining from one to four Legendre polynomials in the seried for the displacement and taking account of from zero to three pairs of St.Venant edge effects. The solution appears in the form of an expansion in numerical powers of $a$, having essentially different (more than 100) coefficients of which the grater number vary little with increase in the number of terms used in the Legendre polynomials. Three significant conclusions can be drawn from the numerical results.

a) It is necessary to proceed with care in the application of asymptotic formulas ($a \to 0$) for the calculation of real plates with finite thickness $a$. In the case considered, the errors in the bending moment of the Kirchhoff theory have values in the asymptotic sense of the order of $a$, $a^2$,... Nevertheless, for $a$ = const > $^1/_{25}$ the numerical values of the "second corrections" are greater than the first! In a part of the bend (outside edge effect zones) the "second corrections" up to $a$ = const > $^1/_{100}$ dominate.

b) Numerical values of the corrections to the Kirchhoff theory are comparatively small even for fairly thick plates. For example, the correction to the bending moment in the middle of the span for $a = {}^1/_3$ is about 12%, and for $a = {}^1/_{10}$, about 1.4%.

c) The conclusion in [8] relating to the accuracy of the Reissner theory must be commented upon. Although the Reissner theory does not determine corrections of the order of $a$ to the Kirchhoff theory in the given case, it may (for appreciable values of $a$) define more accurately the basic state of stress in a plate.

We note that in [9] and [10] the stresses were expanded in a series of Legendre polynomials originating directly from the three-dimensional theory of elasticity. Below, the displacements are expanded in Legendre polynomials for known particular solutions [2] and [8] of the basic state of stress with with edge effects.

**1. Basic notations.** Let $E$ be the modulus of elasticity, $\mu$ Poisson's ratio, $2h$ the plate thickness, $l$ a characteristic dimension of the middle surface, $a = 2h/l$ the relative plate thickness, $\xi,\eta$ and $\zeta$ dimensionless Cartesian coordinates (devided by $h$) of which $\xi$ and $\eta$ are chosen on the middle surface of the plate ($\zeta = 0$), $u_i$ ($i = 1,2,3$) dimensionless displacements (devided by $h$) in the $\xi,\eta$ and $\zeta$ directions respectively, $\sigma_{ij}$ ($i,j = 1,2,3$) dimensionless stresses (multiplied by $(1+\mu)E^{-1}$), $M_{rs}$ and $Q_s$ ($r,s = 1,2$) dimensionless moments and transverse forces, $P_{2m}(\zeta)$ and $P_{2m+1}(\zeta)$ ($m = 0,1,2,\ldots$) Legendre polynomials. We have

$$M_{rs} = \int_{-1}^{+1} \sigma_{rs}\zeta\, d\zeta, \qquad Q_s = \int_{-1}^{+1} \sigma_{s3}\, d\zeta \qquad (r, s = 1, 2) \tag{1.1}$$

$$P_0(\zeta) = 1, \qquad P_1(\zeta) = \zeta, \qquad P_2(\zeta) = {}^1/_2(3\zeta^2 - 1), \ldots \tag{1.2}$$

We denote by $q_j, \bar{q}_j$ the conjugate roots [2] and [8] of Equation

$$\sin 2q = 2q \tag{1.3}$$

which have the property $\operatorname{Re} q_j < 0$. Further, let

$$\begin{aligned} g_j &= \left[\left(-\frac{\zeta q \cos q\zeta}{2-2\mu} + \frac{\cos^2 q \sin q\zeta}{2-2\mu} - \sin q\zeta\right)\frac{q}{\sin q}\right]_{q=q_j} \\ f_j &= \left[\left(\frac{\zeta q \sin q\zeta}{2-2\mu} - \frac{\sin^2 q \cos q\zeta}{2-2\mu} + \cos q\zeta\right)\frac{q}{\sin q}\right]_{q=q_j} \end{aligned} \tag{1.4}$$

Conjugate values $\bar{g}_j, \bar{f}_j$ are obtained from (1.4) by substitution of $\bar{q}_j$ for $q_j$.

In order to describe edge effects near a surface $\xi$ = const we use a coordinate $\xi_*$ directed from the edge in the interior of the plate.

**2. Calculation of a strip.** Passage to this case consists in the expansion (at the edge surfaces) of those quantities which specify the edge conditions in a series of $P$-functions, and in their approximate satisfaction with a finite number of terms of the series. We consider the strip to be infinite along $\eta$, and to have a span $l$. We place the origin of the $\xi$ coordinate in the middle of the span so that on the edges

$$\xi = \pm \xi_0, \qquad \xi_* = 0, \qquad \xi_0 = \frac{l}{2h} = a^{-1} \tag{2.1}$$

Let the strip be loaded by a normal force so that on the surfaces $\zeta = \pm 1$

$$\sigma_{13}(\xi, \pm 1) = \sigma_{23}(\xi, \pm 1) = 0, \qquad \sigma_{33}(\xi, \pm 1) = \pm p, \qquad p = \text{const} \tag{2.2}$$

Here the state of stress depends on the coordinates $\xi$ and $\zeta$ and may be represented [8] as a sum of the basic state of stress $(u_1^{(0)}, u_3^{(0)})$ and the edge effects $(u_1^{(1)}, u_3^{(1)})$, connected with the roots of Equation (1.3). Thus

$$u_1 = u_1^{(0)} + u_1^{(1)}, \qquad u_2 = 0, \qquad u_3 = u_3^{(0)} + u_3^{(1)} \tag{2.3}$$

We assume identical edge conditions on the edges $\xi = \xi_0$ and $\xi = -\xi_0$. By making use of the condition of symmetry, from [8] we obtain Formulas for the displacements

a) for the basic state of stress

$$u_1^{(0)} = (-2\xi A_3 + U_1) P_1 + U_3 P_3 \tag{2.4}$$

$$u_3^{(0)} = \left[A_1 + \left(\xi^2 - \frac{6-4\mu}{3-3\mu}\right) A_3 + U_0\right] P_0 + \left[\frac{2}{3}\frac{\mu}{1-\mu} A_3 + U_2\right] P_2 + U_4 P_4$$

$$U_0 = p\left(\frac{1-\mu}{8}\xi^4 - \frac{3-2\mu}{2}\xi^2 + \frac{3-2\mu}{5}\right), \qquad U_1 = p\xi\left(-\frac{1-\mu}{2}\xi^2 + \frac{3-9\mu}{5}\right)$$

$$U_2 = p\left(\frac{\mu}{2}\xi^2 + \frac{3-4\mu}{7}\right), \qquad U_3 = p\frac{2-\mu}{5}\xi, \qquad U_4 = -p\frac{1+\mu}{35} \tag{2.5}$$

$$M_{11}^{(0)} = p\left(-\xi^2 + \frac{2}{5}\right) - \frac{4}{3-3\mu}A_3, \qquad Q_1^{(0)} = -2p\xi \tag{2.6}$$

Here $A_1$ and $A_3$ are consonants of integration.

b) for the state of stress associated with edge effects, assuming that $l$ is sufficiently large for these effects to be considered separately, i.e.

$$\max(\exp q_j\xi_0) \sim \exp(-3.749\xi_0) \ll 1 \tag{2.7}$$

we have Formulas [8]

$$u_r^{(1)} = \sum_{j=1}^{\infty} (u_{rj}^{(1)} + \bar{u}_{rj}^{(1)}), \qquad u_{1j}^{(1)} = g_j C_j e^{q_j \xi_*}, \qquad u_{3j}^{(1)} = f_j C_j e^{q_j \xi_*} \qquad (r = 1, 3) \tag{2.8}$$

Here $\bar{u}_{rj}^{(1)}$ is the conjugate of $u_{rj}^{(1)}$ and $C_j$, $\bar{C}_j$ are mutually conjugate integration constants. The representation of $u_1^{(1)}$, $u_3^{(1)}$ at $\xi_* = 0$ (on the edge surfaces) leads to an expansion of the function (1.4) in a series of $P$-functions

$$g_j = \sum_{m=0}^{\infty} (G_{j,2m+1} + iG^*_{j,2m+1}) P_{2m+1}, \qquad f_j = \sum_{m=0}^{\infty} (F_{j2m} + iF^*_{j2m}) P_{2m} \tag{2.9}$$

where $G_{j,2m+1}$, $G^*_{j,2m+1}$, $F_{j2m}$, $F^*_{j2m}$ are real numerical coefficients. Their values were found for $\mu = 0.3$; $j = 1,2,3$ and $m = 0,1,2,3$.

It would not be too difficult to write formulas for the stresses in the form of (2.4) and (2.8) as well. It is interesting to note that the expansion of the edge effect stresses in a series of $P$-functions gives zero as the generalized Fourier coefficients for $P_0$ and $P_1$. This means that

$$M_{11}^{(1)} = Q_1^{(1)} = 0 \tag{2.10}$$

E x a m p l e . We set the boundary conditions as

$$u_1 = u_3 = 0 \quad \text{при } \xi = \pm\xi_0 \tag{2.11}$$

and take $\mu = 0.3$. By equating to zero the generalized Fourier coefficients in the expansion of the displacement in $P$-functions, we obtain from (2.11) an infinite system of algebraic equations in respect of $A_1$, $A_3$, $X_j$, $Y_j$ ($j = 1,2,3,\ldots$). Here

$$C_j = {}^1\!/_2 X_j + {}^1\!/_2 iY_j, \qquad C_j = {}^1\!/_2 X_j - {}^1\!/_2 iY_j \tag{2.12}$$

Let these equations be numbered according to the $P$-functions index. Then

Equation zero has zero coefficients for $Y_1$, Equations 1 and 2 have zero coefficients for $A_1$, and all succeeding equations for $A_1$ and $A_3$. The free terms $U_k(\xi_0)$ differ from zero only for the five first equations ($k$ = 0,1,2, 3,4) and are determined from (2.5) by setting $\xi = \xi_0$.

TABLE 1.

| Approximation | Quantity | Coefficients for | | | | |
|---|---|---|---|---|---|---|
| | | $p\xi_0^4$ | $p\xi_0^3$ | $p\xi_0^2$ | $p\xi_0$ | $p$ |
| 1 | $A_1$ | 0.0875 | — | 0.7700 | — | —0.4143 |
| | $A_3$ | — | — | —0.1750 | — | 0.0300 |
| 2 | $A_1$ | 0.0875 | 0.0056 | 0.7752 | 0.0629 | —0.3506 |
| | $A_3$ | — | — | —0.1750 | —0.0056 | 0.0395 |
| | $X_1$ | — | — | —0.0147 | —0.0607 | —0.0395 |
| | $Y_1$ | — | — | 0.0109 | —0.0083 | 0.0293 |
| 3 | $A_1$ | 0.0875 | 0.0058 | 0.7778 | 0.0749 | —0.3365 |
| | $A_3$ | — | — | —0.1750 | —0.0058 | 0.0423 |
| | $X_1$ | — | — | —0.0169 | —0.0640 | —0.0431 |
| | $Y_1$ | — | — | 0.0070 | —0.0134 | 0.0228 |
| | $X_2$ | — | — | —0.0032 | —0.0076 | —0.0040 |
| | $Y_2$ | — | — | 0.0046 | 0.0013 | 0.0094 |
| 4 | $A_1$ | 0.0875 | 0.0067 | 0.7719 | 0.0745 | —0.3450 |
| | $A_3$ | — | — | —0.1750 | —0.0067 | 0.0426 |
| | $X_1$ | — | — | —0.0127 | —0.0631 | —0.0367 |
| | $Y_1$ | — | — | 0.0085 | —0.0137 | 0.0254 |
| | $X_2$ | — | — | —0.0005 | —0.0070 | —0.0001 |
| | $Y_2$ | — | — | 0.0012 | 0.0012 | 0.0039 |
| | $X_3$ | — | — | —0.0012 | —0.0016 | —0.0011 |
| | $Y_3$ | — | — | 0.0027 | —0.0002 | 0.0046 |

We call the first approximation ($n$ = 1) the solution obtained from the first two equations retaining only the unknowns $A_1$ and $A_3$; then the second approximation ($n$ = 2) is the solution obtained from the first four equations retaining the unknowns $A_1$, $A_3$, $X_1$, $Y_1$, etc. Table 1 gives the solution to the first, second, third and forth approximation ($n$=1,2,3,4). The remaining terms of the form

$$\frac{pc}{\xi_0 + d} \qquad (c < 0.05,\ d < 0.02) \tag{2.13}$$

are omitted.

If in the expressions for the first approximation for $A_1$ and $A_3$, only the principal terms are retained (the largest powers of $\xi_0$), then the Kirchhoff solution is obtained. Table 2 shows, in percent, the corrections to the bending moment in the middle of the strip span according to the Kirchhoff theory. The "first" and "second" corrections are

TABLE 2.

| $\alpha$ | $n$ | Corrections to the bending moment in the middle of the strip span of Kirchhoff theory (in %) | | |
|---|---|---|---|---|
| | | complete | "first" order | "second" order |
| $\frac{1}{3}$ | 1 | 11.4 | — | 11.4 |
| | 2 | 12.2 | 1.0 | 10.8 |
| | 3 | 12.1 | 1.1 | 10.6 |
| | 4 | 12.3 | 1.3 | 10.6 |
| $\frac{1}{10}$ | 1 | 1.03 | — | 1.03 |
| | 2 | 1.30 | 0.32 | 0.97 |
| | 3 | 1.30 | 0.33 | 0.96 |
| | 4 | 1.35 | 0.38 | 0.96 |

those corresponding to whether the quantity is of the order of $a$ or $a^2$ as $a \to 0$. The conclusions from Tables 1 and 2 concerning the basic state of stress were given in the introduction. In the edge effect zones the lower approximations $(n \leqslant 4)$ considered, do not guarantee great accuracy, but for all that it is seen that the stress corrections found by Kirchhoff theory are not large numerically, although in the asymptotic sense they are of the order of $a^{(0)} \sim 1$ (for $\xi_0 = a^{-1} \to \infty$).

N o t e. For application of the method described to problems where the state of stress varies also with $\eta$, the three edge conditions must be formulated and the displacements determined in the form of sum

$$u_i = u_i^{(0)} + u_i^{(1)} + u_i^{(2)} \qquad (i = 1, 2, 3)$$

where $u_i^{(0)}$, $u_i^{(1)}$ denote displacements of the basic state of stress and of the edge effects connected with the roots of Equation (1.3), and the $u_i^{(2)}$ are the edge effect displacements connected with the roots of Equation $\cos \lambda = 0$ [8] and having real coefficients $Z_j$. The displacements $u_i^{(2)}$ may likewise be expanded in a series of $P$-functions, then the approximate solution for the edge effect may be obtained analogously to the above with the $Z_j$ as additional unknowns.

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